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Some properties of mixing repellers

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Abstract. Applying the thermodynamic formalism to mixing repellers for regular maps, we obtain some rigorous relations between the dimensions of the repellers and the dynamic properties of the transformation on them. For a class of one-dimensional expanding maps, these quantities can be approximated to any desired accuracy by means of the corresponding variables for linear expanding maps, which can be computed exactly. Finally, a mathematical foundation and some properties of the generalised dimensions are given.

1. Introduction

In the last few years several authors have considered the ergodic properties of a transformation on a repeller in connection with its geometric structure [1-12]. If the transformation is sufficiently regular in a neighbourhood of the repeller and uniformly hyperbolic on it (mixing repeller), one can apply the powerful techniques of the axiom A systems to understand the dynamics on the repeller and its fractal properties. In this paper we especially consider a conformal transformation on the repeller, i.e. in every point the tangent map is a scalar times an isometry. Sullivan [8] has already pointed out the importance of these systems which include all the one-dimensional differentiable maps, the rational endomorphisms of the Riemann sphere and every group of Möbius transformations of the n-sphere.

In this paper we get the following results.

(i) We bound from above the Hausdorff dimension of any ergodic measure on a mixing repeller by means of the Kolmogorov (metric) entropy and the lowest Lyapunov exponent. The existence of a similar lower bound in terms of the greatest Lyapunov exponent makes the Hausdorff dimension of the measure, when the transformation is conformal, equal to the ratio of the metric entropy and the unique Lyapunov exponent. This result already exists for analytic maps [13, 14] and for one-dimensional maps [15], but it has been proved with different techniques. Our method is based on proposition 2.1 in Young [16] and gives an easy proof in the general case of conformal transformations.

(ii) As already shown by Eckmann and Ruelle [17] for the almost attractors, also for mixing repellers, the theoretic escape rate defined by Kadanoff and Tang [6] is simply related to the topological pressure. Combining this result with (i) we immediately get some rigorous relations for the escape rate, the Hausdorff dimension of the measure and the Lyapunov exponent, which have already been obtained by Kantz and Grassberger [2] and Takesue [3] using heuristic arguments.

(iii) For a simple one-dimensional mixing repeller (the linear Cantor set) we perform the easy computation of the pressure and, following an approximation procedure recently proposed [18, 19], we extend it, by a convergence theorem, to each disconnected hyperbolic repeller in dimension one. This allows us to get the escape rate and the Hausdorff dimension to any desired accuracy, and then to give a meaning to the singularity of the double energy integral with respect to the maximal entropy measure: this has intrinsic interest in potential theory.

(iv) We show that for conformal mixing repellers, and in general for all the systems where Young's theorem (theorem 4.4 in [16]) applies, the generalised dimensions introduced in [20] are all equal on infinitely many subsets of full measure.

2. Hausdorff dimensions of mixing repellers

We adopt the following definition of mixing repeller (see, for instance, [1]).

Definition 2.1. Let M be a compact and connected finite-dimensional Riemannian manifold of class C^∞, $T: M \rightleftarrows M$ a sufficiently regular transformation (see below) and J a compact subset of M. We say that J is a mixing repeller for T if there exists an open subset $V \supset J$ such that the following is true.

(i) $T_{|V}$ is of class $C^{1+\varepsilon}$ ($\varepsilon > 0$). (In the following proposition it will be sufficient to take T of class C^1. The stronger regularity is needed for some results quoted in the next sections.) Also $\bigcap_{n=0}^{\infty} T^{-n}V = J$. ($T^{-1}\Omega$ or $T^{-1}[\Omega]$ denote the inverse image of Ω.)

(ii) If U is an open set intersecting J, there exists an $n > 0$ such that $T^n U \supset J$ (condition of topological mixing).

(iii) T is uniformly expanding on J, i.e. $\exists 0 < c < 1$ and $\gamma > 1$:

$$\|(D_x T^n)u\| \geqslant c\gamma^n \|u\| \tag{2.1}$$

$\forall x \in J$, $u \in T_x M$ and $n \geqslant 1$. Here $(D_x T)u$ (also denoted as $DT(x)u$) is the tangent map of T in $x \in M$ applied to a vector u of the tangent space of M in x and $\| \ \|$ is the norm on the tangent bundle with respect to a smooth Riemann metric. It is well known that it is possible to choose an adapted smooth Riemann metric for which $c = 1$ in (2.1): in the following we always refer to this metric. By condition (2.1) the Lyapunov exponents of any T-invariant and ergodic probability measure μ on J are positive and therefore the Kolmogorov entropy $h(\mu)$ is finite. We denote by $\lambda_{\mu,\max}$ and $\lambda_{\mu,\min}$, respectively, the largest and smallest Lyapunov exponent of the measure μ and with

$$\mathrm{HD}(\mu) = \inf_{\substack{Y \subset J \\ \mu(Y)=1}} \{\text{Hausdorff dimension of } Y\}$$

the Hausdorff dimension of the measure μ.

Theorem 2.2. Let $T: V \supset J \mapsto M$ be a transformation of class C^1 and μ a T-invariant and ergodic non-atomic probability measure on the Borel subsets of J. Then

$$\frac{h(\mu)}{\lambda_{\mu,\max}} \leqslant \mathrm{HD}(\mu) \leqslant \frac{h(\mu)}{\lambda_{\mu,\min}}. \tag{2.2}$$

When the transformation is conformal, the μ Lyapunov exponent λ_μ is unique and we clearly have

$$\mathrm{HD}(\mu) = h(\mu)/\lambda_\mu. \tag{2.3}$$

Proof. In the proof of the theorem only properties (i) and (iii) of the mixing repellers are used; moreover, our results remain true if V is a bounded subset of $\mathbb{R}^n$. By the very definition of the Hausdorff dimension of the measure and by proposition 2.1 in Young [16], it will be sufficient to show that μ almost everywhere $x \in J$:

$$\frac{h(\mu)}{\lambda_{\mu,\max}} \leq \liminf_{l \to 0^+} \frac{\log \mu(B(x,l) \cap J)}{\log l}$$
$$\leq \limsup_{l \to 0^+} \frac{\log \mu(B(x,l) \cap J)}{\log l} \leq \frac{h(\mu)}{\lambda_{\mu,\min}}$$

where $B(x, l)$ is the closed ball (in the metric on M associated with the Riemannian structure) of centre x and radius l. The lower bound was essentially proved in Ledrappier [21]; it is only necessary to use the Brin–Katok theorem [22] instead of the local characterisation of the metric entropy given in [21]. Now we give the upper bound. First of all we replace the continuous variable l with a discrete sequence l_n such that $l_n \to 0$ and $\log l_n / \log l_{n+1} \to 1$ when $n \to +\infty$. By hypothesis and the easy arguments of compactness, we can find a finite cover of J by sufficiently small open balls $\{F_m\}_{m=1}^{M}$ such that $\|D_x T\| \neq 0$ when†

$$x \in \bigcup_{m=1}^{M} \overline{F_m} = F \subset V$$

and

$$T^{-1}(\overline{F_m}) = \bigcap_{j=1}^{s_m} P_{j,m}$$

is the disjoint union of $s_m < +\infty$ (closed) sets, on each of which the mapping T is a diffeomorphism: we denote by $T_{j,m}^{-1}$ the inverse of the restriction of T to $P_{j,m}$. Since the functions $\log\|[DT(T_{j,m}^{-1}(x))]^{-1}\|$ are uniformly continuous on $\overline{F_m}$, for every $\varepsilon > 0$, there exists a $\nu > 0$ and another finite cover of J by open convex balls $\{G_i\}_{i=1}^{I}$ of diameter less than ν and such that each of their closures is included in one of the F_m, for which

$$\|[DT(T_{j,m}^{-1}(x))]^{-1}\| \leq e^{\varepsilon} \|[DT(T_{j,m}^{-1}(y))]^{-1}\| \tag{2.4}$$

when x and y belong to a certain G_i and for every $1 \leq m \leq M$, $1 \leq j \leq s_m$. We call $\chi(\varepsilon)$ the Lebesgue number of this partition. Then we take a finite Borel partition $\mathscr{B}$ of J of diameter less than $\chi(\varepsilon)$. Making this the atom of the partition $\bigvee_{k=0}^{n-1} T^{-k}\mathscr{B}$ which contains $x \in J$, $B_n(x)$ will be included in a closed ball $B(x, l_n(x))$ whose radius will satisfy

$$\lim_{n \to +\infty} \left(-\frac{1}{n} \log l_n(x) \right) = -\int_J \log \|(D_x T)^{-1}\| \, d\mu - \varepsilon$$

μ almost everywhere x. Then applying the Shannon–McMillan theorem [23] we get μ almost everywhere x

$$\limsup_{n \to +\infty} \frac{\log \mu[B(x, l_n(x))]}{\log l_n(x)}$$
$$\leq \limsup_{n \to +\infty} \left(-\frac{1}{n} \log \mu(B_n(x)) \right) \left(-\int_J \log \|(D_x T)^{-1}\| \, d\mu - \varepsilon \right)^{-1}$$
$$\leq \frac{h(\mu)}{-\int_J \log \|(D_x T)^{-1}\| \, d\mu}$$

† By $\bar{A}$ we denote the closure of A.

by the arbitrariness of ε. Since we can apply the above inequality to each power T^p of the map, $p>1$, by the multiplicative ergodic theorem [24], $(1/p)\log\|(D_xT^p)^{-1}\|$ converges μ almost everywhere x to the reciprocal of the smallest μ Lyapunov exponent, when $p\to+\infty$, and this proves (2.2). To finish the proof, we take an element $B\in\mathscr{B}$ and observe that it belongs to a certain G_i and $\overline{G_i}\subset F_m$ with $1\leq m\leq M$. By condition (2.4) and the invariance of T on J we have, for every $1\leq j\leq s_m$,

$$\operatorname{diam}(T_{j,m}^{-1}B)\leq\max_{y\in\bar{G}_i}\|DT_{j,m}^{-1}(y)\|\operatorname{diam}B\leq\|[DT(T_{j,m}^{-1}(x))]^{-1}\|\mathrm{e}^{\varepsilon}\chi(\varepsilon)\leq\frac{e^{\varepsilon}\chi(\varepsilon)}{\gamma}\leq\chi(\varepsilon)$$

where $x\in B$ and with the assumption that $\varepsilon<\log\gamma$. By induction it is now easy to prove that $\forall n\geq 1$ and when the composition of functions, expressed by $\boxed{\circ}$, makes sense, we have

$$\operatorname{diam}\left[\left(\boxed{\circ}_{j=1}^{n}T_{i_j,\eta_j}^{-1}\right)B\right]$$

$$\leq\mathrm{e}^{n\varepsilon}\chi(\varepsilon)\prod_{j=1}^{n}\left\|\left[DT\left(\left(\boxed{\circ}_{t=1}^{j}T_{i_t,\eta_t}^{-1}\right)(x)\right)\right]^{-1}\right\|$$

$$=\mathrm{e}^{n\varepsilon}\chi(\varepsilon)\prod_{j=0}^{n-1}\|[DT(T^j(x'))]^{-1}\|\leq\chi(\varepsilon)$$

where x' is any point in $(\boxed{\circ}_{j=1}^{n}T_{i_j,\eta_j}^{-1})B$; $i_j\in(1,\ldots,s_{\eta_j})$ and $\eta_j\in(1,\ldots,M)$.

If we now take a point $x\in J$, $B_n(x)$ is included in an element of the partition $T^{-(n-1)}\mathscr{B}$. Therefore, by the above considerations, it is sufficient to take μ almost everywhere x

$$l_n(x)=\exp[(n-1)\varepsilon]\chi(\varepsilon)\prod_{j=0}^{n-2}\|[DT(T^j(x))]^{-1}\|$$

for $\varepsilon<\log\gamma$ and $n>2$.

Remark 2.3. If the mapping has degree s, and if we consider the maximal entropy measure on J, we have $h(\mu)=\log s$. Besides, $\lambda_{\mu,\max}$ and $\lambda_{\mu,\min}$ can sometimes be replaced, respectively, with $\max_{x\in J}\log\|D_xT\|$ and $\min_{x\in J}\log\|(D_xT)^{-1}\|^{-1}$, which are easier to compute and the upper bound remains strictly smaller than the dimension of the manifold (here $(D_xT)^{-1}$ denotes the inverse of the tangent map in x). In § 3 we give an easy example where these considerations apply. A result parallel to (2.2) for a diffeomorphism of a compact Riemannian manifold is given in [16].

3. Thermodynamical properties of mixing repellers

For conformal transformations of class $C^{1+\varepsilon}$, the Hausdorff dimension d_{H} of J is determined by the Bowen–Ruelle formula [1]:

$$P(T,-d_{\mathrm{H}}\log\|D_xT\|)=\max\left\{h(\mu)-d_{\mathrm{H}}\int_J\log\|D_xT\|\,\mathrm{d}\mu(x)\right\}=0 \tag{3.1}$$

where the maximum is taken on the set of T-invariant probability Borel measures on J, $M_T(J)$, and P is the (topological) pressure [25, 26]. The maximum has attained, for a unique ergodic Borel measure, the equilibrium measure μ_E for $-d_H \log\|D_xT\|$. Later on we will use a general definition of the pressure and its specialisation to topologically mixing dynamical systems. In the first case, let $T: J \to J$ be a continuous function of a compact metric space J into itself and φ an element of the Banach space $C(J, \mathbb{R})$ of the real-valued continuous functions on J equipped with the C^∞ norm. Then let $\mathscr{A}^0$ be an open cover of J and diam $\mathscr{A}^0$ its diameter. Following Walters [26] we define

$$P_n(T, \varphi, \mathscr{A}^0) = \inf\left\{\sum_{A\in\alpha} \sup_{x\in A} \exp\left(\sum_{i=0}^{n-1} \varphi(T^i(x))\right);\ \alpha\text{: finite subcover of } \mathscr{A}^{n-1}\right\}$$

with

$$\mathscr{A}^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\mathscr{A}^0.$$

Then the thermodynamic limit:

$$P(T, \varphi, \mathscr{A}^0) = \lim_{n\to+\infty} \frac{1}{n} \log P_n(T, \varphi, \mathscr{A}^0)$$

exists and finally the pressure of the function φ is defined as

$$P(T, \varphi) = \lim_{\delta\to 0} [\sup\{P(T, \varphi, \mathscr{A}^0) | \text{diam } \mathscr{A}^0 < \delta\}].$$

We recall that, in general, the pressure is given by the variational principle [25, 26]:

$$P(T, \varphi) = \sup_{\mu\in M_T(J)} \left\{h(\mu) + \int_J \varphi(x)\, \mathrm{d}\mu(x)\right\}. \tag{3.2}$$

If T is topologically mixing on J, the functions $\tilde{P}_n$ defined on $\varphi \in C(J, \mathbb{R})$ as

$$\tilde{P}_n(\varphi) = \frac{1}{n} \log \sum_{x\in \text{Fix}\, T^n} \exp\left(\sum_{k=0}^{n-1} \varphi(T^k(x))\right) \tag{3.3}$$

tend pointwise to $P(T, \varphi)$ when $n \to +\infty$. We recall that the condition of topological mixing implies that the periodic points of T^n, Fix T^n, are dense in J (see [25], ch 7, example 4). Moreover the equilibrium measure for φ, i.e. the unique T-invariant and ergodic probability measure which realises the maximum in (3.2), is the weak limit of the sequence of the T-invariant normalised measures $\mu^{(n)}$ with support in Fix T^n, given by

$$\mu^{(n)}\{z\} = \exp\left(\sum_{k=0}^{n-1} \varphi(T^k(z))\right)\left(\sum_{x\in \text{Fix}\, T^n} \exp \sum_{k=0}^{n-1} \varphi(T^k(x))\right)^{-1} \tag{3.4}$$

where $z \in \text{Fix}\, T^n$. For the derivation of formulae (3.3) and (3.4) see [25], § 7.28. Before going on, we also have to recall the definition of the escape rates. Let us take a neighbourhood U of the repeller J and spray uniformly on it (with respect to the Riemannian measure on $M \supset J$) a large number of points N_0. After n iterations and for large n the number of points in U, N_n, decays exponentially as $(N_n/N_0) \sim q^{-n}$ [6].

We call $\rho_E = \log q$ the experimental escape rate. For the hyperbolic repelling sets, Kadanoff and Tang [6, 7] have conjectured that ρ_E is the same as the theoretical escape rate ρ_T defined as

$$\rho_T = \lim_{n\to+\infty}\left[-\frac{1}{n}\log\left(\sum_{x\in \mathrm{Fix}\, T^n}\frac{1}{|\det(\mathbb{1}-D_xT^n)|}\right)\right]. \tag{3.5}$$

This definition is formally derived by the ergodic properties of the axiom A systems; another characterisation will be useful. Indeed, the expanding condition (2.1) implies that the tangent map: $D_xT^n: T_xM \to T_{T_x^n}M$, $x\in J$ is invertible for $n\geq 1$ and moreover $|\det(\mathbb{1}-D_xT^n)|\to|\det D_xT^n|$ when $n\to\infty$ and uniformly on J. Using this fact in (3.5) and computing the pressure of $(-\log|\det D_xT|)$ by means of (3.3) we immediately get

$$\rho_T = -P(T, -\log|\det D_xT|). \tag{3.6}$$

Now we consider the equilibrium measure $\bar\mu_E$ for the function $-\log|\det D_xT|$. Since $\bar\mu_E$ is T-ergodic we have

$$\int_J \log|\det D_xT|\,\mathrm{d}\bar\mu_E(x) = \sum_i \zeta_i\lambda_{\bar\mu_E,i}$$

where $\lambda_{\bar\mu_E,i}$ denotes the ith Lyapunov exponent of multiplicity ζ_i, $\Sigma_i\zeta_i = d$, d being the dimension of the manifold M. Applying this to the variational principle (3.2) we obtain

$$\rho_T = \sum_i \zeta_i\lambda_{\bar\mu_E,i} - h(\bar\mu_E)$$

which, for conformal transformations, becomes simply

$$\rho_T = d\lambda_{\bar\mu_E} - h(\bar\mu_E). \tag{3.7}$$

In the following we limit ourselves to these transformations. For every other T-ergodic measure μ, (3.7) gives

$$\rho_T \leq d\lambda_\mu - h(\mu). \tag{3.8}$$

Therefore, by (2.3)

$$\mathrm{HD}(\mu) \leq d - \rho_T/\lambda_\mu. \tag{3.9}$$

The equality in (3.9) is attained for the equilibrium measure $\bar\mu_E$ which, as we have already said, is the weak limit of the measures given by (3.4) with $\varphi(x) = -d\log\|D_xT\|$. These measures coincide exactly with the sequence of probability measures proposed with heuristic arguments by Kantz and Grassberger [2].

They call $\bar\mu_E$ the 'natural' measure for the repeller; as pointed out by the same authors, this natural measure does not coincide, in general, with the measure of maximal entropy (or balanced measure, see below) for the repeller. As we shall see soon, this second measure is particularly useful in computing the dynamical variables of the repeller, since it can be reconstructed by a 'time series' starting from a point in a close neighbourhood to the repeller itself. In this sense, the balanced measure is much more 'natural' than the preceding one. Now we go back to formula (3.1) which allows us to determine the Hausdorff dimension d_H of the repeller. If we call μ_E the equilibrium measure for $(-d_H\log\|D_xT\|)$, since $d_H = h(\mu_E)/\lambda_{\mu E}$, inserting it in (3.8) we obtain

$$d_H \leq \frac{dh(\mu_E)}{\rho_T + h(\mu_E)} \leq \frac{dh(\mu_E)}{h(\mu_E)(\rho_T/h(\mu_E)+1)}.$$

Since the topological entropy h_{TOP} maximises the metric entropies, substituting it in place of $h(\mu_E)$ we obtain the following proposition.

Proposition 3.1. For a conformal mixing repeller in a compact Riemannian manifold of dimension d, the Hausdorff dimension satisfies the bound:

$$d_{\mathrm{H}} \leqslant \frac{d h_{\mathrm{top}}}{\rho_{\mathrm{T}} + h_{\mathrm{top}}}. \tag{3.10}$$

This inequality generalises to every dimension d the same relation obtained, in a non-rigorous way, by Takesue [3] for one-dimensional maps.

4. Linear Cantorian approximation

In this section, we develop the approximation scheme proposed in [18, 19] and apply the results of §§ 2 and 3 to a simple class of one-dimensional mixing repellers, already studied by Pelikan [11]. They are the invariant sets of maps T from an open neighbourhood V of $[0, 1]$ into $\mathbb{R}$ and of class C^2 on V, with the following properties:

(i) $T^{-1}[0, 1] \subset [0, 1]$,

(ii) T is expanding on $T^{-1}[0, 1]$, i.e. $|T'(x)| \geqslant \gamma > 1$ for $x \in T^{-1}[0, 1]$,

(iii) $T^{-1}[0, 1]$ is the disjoint union of s (closed) intervals.

With these prescriptions the repeller

$$J = \bigcap_{k=0}^{\infty} T^{-k}[0, 1]$$

is a completely invariant Cantor set.

We call these maps 'Cantorian maps with s inverses'. The s^n pairwise disjoint sets $A_{j,n} \cap J$, $j = 1, \ldots, s^n$, where $A_{j,n}$ belongs to $T^{-n}[0, 1]$, form a cover of J. We denote by T_j^{-n} the inverse of the restriction of T^n to $A_{j,n}$ and call $\{T_J^{-n}\}_1^{s^n}$ the inverse determinations of T^n. For an easy distortion argument, see, for example, [11], there is a constant $g \geqslant 1$ such that, for every pair of points (x, y) in the same $A_{j,n}$ and for every $n > 0$, we have

$$g^{-1}|(T^n)'(y)| \leqslant |(T^n)'(x)| \leqslant g|(T^n)'(y)|. \tag{4.1}$$

Now we need the following.

Theorem 4.1. There exists a unique probability measure μ_{B} on the Borel subsets of $[0, 1]$ with the following properties.

(i) μ_{B} is supported on J.

(ii) For every continuous function f on $[0, 1]$ and any point $x_0 \in [0, 1]$ we have, uniformly on $[0, 1]$,

$$\lim_{k \to \infty} \left(\frac{1}{s}\right)^k \sum_{\xi \in T^{-k}\{x_0\}} f(\xi) = \int_J f(x)\, \mathrm{d}\mu_{\mathrm{B}}(x). \tag{4.2}$$

(iii) For every Borel set $A \subset [0, 1]$ where T is injective we have $\mu_{\mathrm{B}}(A) = (1/s)\mu_{\mathrm{B}}(TA)$.

(iv) μ_{B} is T invariant and ergodic; moreover $h(\mu_{\mathrm{B}}) = \log s =$ topological entropy. We call μ_{B} the 'balanced measure' on J.

Outline of the proof. The proof repeats word for word the proof of the existence and unicity of the balanced measure for rational endomorphisms of the Riemann sphere given by Ljubich [27]; it is greatly simplified because the map is uniformly expanding.

A typical condition of the rational maps that is crucial in the proof, i.e. the rational homogeneity of the Julia set [28], also holds for the repeller J. In fact, for every open set 0 containing a point of J, we can find an integer $n>0$ such that $T^n 0 \supset [0,1] \supset J$ (topological mixing). If this were not true, there would be at least one point $p \in [0,1]$ such that $0 \cap T^{-m}\{p\} = \varnothing$, $\forall m>0$. This is impossible, since the set $\{T^{-m}\{p\}\}_{m=1}^{\infty}$, $p \in [0,1]$, is dense in J (see, for example, [12], theorem 3).

A particular family of Cantorian mappings $L: V \to \mathbb{R}$ plays a special role for our further considerations: they are piecewise linear on $L^{-1}[0,1]$ and we denote with λ_i^{-1}, $i=1,\ldots,s$, the absolute values of their slopes. We call their invariant repeller J linear Cantor set with s scales $(\lambda_1,\ldots,\lambda_s)$†.

Lemma 4.2. If L is a linear Cantorian mapping with scales $\lambda_1,\ldots,\lambda_s$, the pressure of the function $\delta \log|L'(z)|$ with L and L' restricted to J and $\delta \in \mathbb{R}$ is

$$P(L, \delta \log|T'(z)|) = \log(\lambda_1^{-\delta} + \ldots + \lambda_s^{-\delta}). \tag{4.3}$$

Proof. We consider the cover of J (both open and closed in the induced topology): $\mathscr{A}^0 = \{A_{i,1} = L_i^{-1}[0,1] \cap J, i=1,\ldots,s\}$. Following the definition and the notation for the pressure given in § 3, we begin to observe that $P(T, \delta\log|L'(z)|) = P(L, \delta\log|L'(z)|, \mathscr{A}^0)$, since $\operatorname{diam}(\mathscr{A}^n) \to 0$ as $n\to\infty$ ([26], theorem 1.9). Moreover, since $\mathscr{A}^k \subset \mathscr{A}^{k-1}$ for every $k>1$ and $\mathscr{A}^{k-1}$ is a minimal open cover of J, we have (the sets $A_{i,k}$, $i=1,\ldots,s^k$, belong to $L^{-k}[0,1] \cap J$):

$$\begin{aligned} P_k(L, \delta\log|L'(z)|, \mathscr{A}^0) &= \sum_{x \in A_{i,k} \in \mathscr{A}^{k-1}} \exp\left(\sum_{j=0}^{k-1} \delta \log|L'(L^j(x))|\right) \\ &= \sum_{i_0=1}^{s} \ldots \sum_{i_{k-1}=1}^{s} \prod_{j=0}^{k-1} \lambda_{i_j}^{-\delta} = (\lambda_1^{-\delta} + \ldots + \lambda_s^{-\delta})^k. \end{aligned}$$

Hence

$$\begin{aligned} P(L, \delta\log|L'(z)|) &= \lim_{k\to\infty} \frac{1}{k} \log P_k(L, \delta\log|L'(z)|, \mathscr{A}^0) \\ &= \log(\lambda_1^{-\delta} + \ldots + \lambda_s^{-\delta}). \end{aligned}$$

By relation (3.6) we can write explicitly the theoretical escape rate for the Cantor set:

$$\rho_{\mathrm{T}} = -\log(\lambda_1 + \ldots + \lambda_s). \tag{4.4}$$

We also recall that the Hausdorff dimension d_{H} of J is determined by the equation [29]

$$\lambda_1^{d_{\mathrm{H}}} + \ldots + \lambda_s^{d_{\mathrm{H}}} = 1 \tag{4.5}$$

and the Lyapunov exponent of the balanced measure μ_{B} is given by (we use here statement (iii) of theorem 4.1)

$$\begin{aligned} \lambda(\mu_{\mathrm{B}}) &= \int_J \log|L'(z)| \, \mathrm{d}\mu_{\mathrm{B}} \\ &= \sum_{i=1}^{s} \int_{T_i^{-1}[J]} \log|L'(z)| \, \mathrm{d}\mu_{\mathrm{B}} = -(1/s)[\log(\lambda_1) + \ldots + \log(\lambda_s)]. \end{aligned} \tag{4.6}$$

† To avoid confusion, we recall that λ with subindex a measure denotes a Lyapunov exponent, while λ with index an integer number is a scale of a linear Cantor.

Before going on, we give an easy example of a non-conformal mixing repeller where the bounds (2.2) apply. Let us consider two Cantor sets J_1 and J_2 generated by linear functions L_1 and L_2, each of which has two scales (q_1^{-1}, q_2^{-1}) and (q_3^{-1}, q_4^{-1}), respectively, ordered as $2<q_1<q_2<q_3<q_4$; we endow them with the balanced measures μ_{B_1} and μ_{B_2}. Then we put $(J_1\times J_2)\subset\mathbb{R}^2$ the cartesian product of the Cantor sets with the measure $\mu=(\mu_{B_1}\times\mu_{B_2})$ on it and the transformation L: $(J_1\times J_2)\to(J_1\times J_2)$ defined as $L(x_1, x_2)=(L_1x_1, L_2x_2)$, $x_1\in J_1$, $x_2\in J_2$. Since $h(\mu)=\log 4$,

$$\max_{X\in J_1\times J_2} \log\|D_xL\| = \log q_4$$

$x=(x_1, x_2)$ and

$$\min_{X\in J_1\times J_2} \log\|(D_xL)^{-1}\|^{-1} = \log q_1$$

by remark 2.3 we conclude that

$$\frac{\log 4}{\log q_4}\leqslant \mathrm{HD}(\mu)\leqslant\frac{\log 4}{\log q_1}.$$

(For similar bounds applied to Julia sets, see [30].)

Besides the Hausdorff dimension, there is another useful index to characterise the fractal properties of an invariant set J. Let μ be a probability measure supported by J. If we define the 'energy integral':

$$\phi_\alpha(\mu)=\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\alpha} \qquad \alpha>0 \tag{4.7}$$

then, by a remarkable theorem of Frostman [29], we have

$$\alpha_\mu=\inf\{\alpha;\,\phi_\alpha(\mu)=+\infty\}\leqslant d_\mathrm{H}(J). \tag{4.8}$$

We call α_μ the divergence abscissa of the measure μ ($d_\mathrm{H}(J)$, or simply d_H, is again the Hausdorff dimension of J). If, for every α, the energy integral is computed with respect to the unique measure which minimises (4.7) (the so-called 'equilibrium measure'; see [31])† the infimum in (4.8) gives exactly the Hausdorff dimension. For some easy models [9, 33], the divergence abscissa was exactly computed with respect to the balanced measure and it was shown that it equals the correlation dimension, introduced by Grassberger and Procaccia [34].

For a linear Cantor set with scales $(\lambda_1, \ldots, \lambda_s)$ we have shown [33] that the divergence abscissa with respect to the balanced measure, which we simply call α, is the real positive solution of the equation:

$$\lambda_1^{-\alpha}+\ldots+\lambda_s^{-\alpha}=s^2. \tag{4.9}$$

Inspecting formulae (4.3) and (4.9) we see that α is the unique real positive root of the equation:

$$P(T, \alpha\log|L'(z)|)=\log s^2. \tag{4.10}$$

This relation shows that the divergence abscissa has a dynamical meaning. This is a novelty in potential theory, where only the equilibrium measures (which are almost always very difficult to construct: see, for example, [32]) carry information about the

† We want to specify that the equilibrium measures of potential theory are, in general, different from the equilibrium measures for the pressure; see [32] for a discussion of this topic.

geometric structure of the fractal. Now we show that (4.10) also holds in a weaker form for non-linear Cantorian mappings with s inverses. This result is part of a more general theorem concerning the approximation of the dynamical and geometric properties of a general one-dimensional repeller by means of linear Cantor sets. The starting point is to associate with each Cantorian mapping T with s inverses, a family $\{L_m\}_{m=1}^{\infty}$ of linear Cantorian mappings in the following way: L_m has s^m scales $\lambda_{m,1},\ldots,\lambda_{m,s^m}$, determined by

$$\lambda_{m,i}=\operatorname{diam}(T_i^{-m}[0,1]) \qquad i=1,\ldots,s^m. \tag{4.11}$$

If J is the repeller of T, we call C_m the repeller of L_m, μ is the balanced measure for T and μ_m is the balanced measure for L_m. Finally if d_{H}, ρ_{T} and α denote, respectively, the Hausdorff dimension, the escape rate and the divergence abscissa for T, d_{H}^m, ρ_{T}^m and α_m will be the corresponding quantities for L_m. In [19], we have shown that $\lim_{m\to\infty}\lambda_{\mu_m}/m=\lambda_\mu$ and then, as an easy consequence, $\lim_{m\to\infty}\mathrm{HD}(\mu_m)=\mathrm{HD}(\mu)$. Moreover we have proved that $\lim_{m\to\infty} d_{\mathrm{H}}^m=d_{\mathrm{H}}$ with geometric fractal arguments. This result is a corollary of the following theorem.

Theorem 4.3. If the maps we are considering are restricted to their repellers we have

$$\lim_{m\to\infty}\frac{1}{m}P(L_m,\delta\log|L_m'(z)|)=P(T,\delta\log|T'(z)|)$$

$\delta\in\mathbb{R}$, and the convergence is uniform in δ on any compact subset of $\mathbb{R}$.

As a consequence we have this corollary.

Corollary 4.4.

(i) $\lim_{m\to\infty}\rho_{\mathrm{T}}^m/m=\rho_{\mathrm{T}}$.

(ii) $\lim_{m\to\infty}\alpha_m=\bar{\alpha}$ and $\bar{\alpha}$ satisfies $P(T,\bar{\alpha}\log|T'(z)|)=\log s^2$ and $\bar{\alpha}\leqslant\mathrm{HD}(\mu)=\log s/\lambda_\mu$.

(iii) $\lim_{m\to\infty} d_{\mathrm{H}}^m=d_{\mathrm{H}}$.

Proof of theorem 4.3 and corollary 4.4. Using the notations, the topological prescriptions and the easy arguments of lemma 4.2 we have, if $\mathscr{A}^{m-1}$ is the cover of J by the sets $\{T_i^{-m}[0,1]\cap J\}$, $i=1,\ldots,s^m$, $m\geqslant 1$:

$$\sum_{i_0=1}^{s^m}\cdots\sum_{i_{k-1}=1}^{s^m}\prod_{J=0}^{k-1}\min_{\xi\in T_{i_J}^{-m}[0,1]}|(T^m)'(\xi)|^\delta\leqslant P_k(T^m,\log|(T^m)'(x)|^\delta,\mathscr{A}^{m-1})$$
$$\leqslant\sum_{i_0=1}^{s^m}\cdots\sum_{i_{k-1}=1}^{s^m}\prod_{J=0}^{k-1}\max_{\xi\in T_{i_J}^{-m}[0,1]}|(T^m)'(\xi)|^\delta.$$

We suppose here that $\delta>0$. If $\delta<0$ we replace max with min. Since every inverse determination of T^m is C^2 on $[0,1]$, there exists a point $\xi_i'\in\operatorname{int}\{T_i^{-m}[0,1]\}$, $i=1,\ldots,s^m$, such that

$$\lambda_{m,i}=|T_i^{-m}(1)-T_i^{-m}(0)|=|(T^m)'(\xi_i')|^{-1}.$$

Using this fact and condition (4.1) we get

$$g^{-\delta k}(\lambda_{m,1}^{-\delta}+\ldots+\lambda_{m,s^m}^{-\delta})^k\leqslant P_k(T^m,\log|(T^m)'(x)|^\delta,\mathscr{A}^{m-1})$$
$$\leqslant g^{\delta k}(\lambda_{m,1}^{-\delta}+\ldots+\lambda_{m,s^m}^{-\delta})^k.$$

We take the logarithm, divide by k and finally set $k \to +\infty$. Thus we get

$$-\delta \log g + \log(\lambda_{m,1}^{-\delta} + \ldots + \lambda_{m,s^m}^{-\delta}) \leq P(T^m, \log|(T^m)'(x)|^\delta, \mathscr{A}^{m-1})$$
$$\leq \delta \log g + \log(\lambda_{m,1}^{-\delta} + \ldots + \lambda_{m,s^m}^{-\delta}).$$

Since diam $\mathscr{A}^{m-1} \to 0$ when $m \to +\infty$ and by theorem 2.2 in [26] we have

$$P(T^m, \log|(T^m)'(x)|^\delta, \mathscr{A}^{m-1}) = P(T^m, \log|(T^m)'(x)|^\delta)$$
$$= mP(T, \log|T'(x)|^\delta).$$

Thus, by lemma 4.2 and for each $\delta \in \mathbb{R}$:

$$|P(T, \log|T'(x)|^\delta) - (1/m)P(L_m, \log|L'_m(x)|^\delta)| \leq |\delta| \log g/m$$

and this proves the theorem.

If we now put $\delta = -1$ and by (3.6) we obtain (i) of the corollary. To show (ii), we begin by noting that

$$\frac{\log s}{\log F} \leq \alpha_m \leq \mathrm{HD}(\mu_m) \leq \frac{\log s}{\log \gamma}$$

where $F = \max_{\xi \in T^{-1}[0,1]} |T'(\xi)|$. This follows by applying Jensen's inequality to equation (4.9) and using, always in (4.9), the obvious condition $(1/F^m) \leq \lambda_{m,i}, \forall i = 1, \ldots, s^m$. By (4.10) and the last inequality of the proof of theorem 4.3, we get

$$|P(T, \alpha_m \log|T'(z)|) - \log s^2|$$
$$= \left| P(T, \alpha_m \log|T'(z)|) - \frac{1}{m} P(L_m, \alpha_m \log|L'_m(x)|) \right|$$
$$\leq \frac{\log s}{\log \gamma} \frac{\log g}{m}.$$

Then

$$\lim_{m \to \infty} P(T, \alpha_m \log|T'(z)|) = \log s^2.$$

Now we prove that the sequence α_m has a limit; it is bounded and if we put $\alpha^- = \liminf_{m\to\infty} \alpha_m$, $\alpha^- \in D$ and $\alpha^+ = \limsup_{m\to\infty} \alpha_m$, $\alpha^+ \in D$, where $D = [\log s/\log F, \log s/\log \gamma]$, there are two subsequences α_{m_k} and α_{m_j} that converge, respectively, to α^- and α^+. By the uniform continuity in α of the pressure on D, we have

$$\lim_{m\to\infty} P(T, \alpha_m \log|T'(x)|) = \lim_{k\to\infty} P(T, \alpha_{m_k} \log|T'(x)|)$$
$$= P(T, \alpha^- \log|T'(x)|) = \log s^2$$

and the same holds if we replace α^- with α^+. But, since $P(T, \delta \log|T'(x)|)$ is injective for $\delta \in \mathbb{R}$, being that $P(L_m, \delta \log|L'_m(x)|)$, we conclude that $\alpha^- = \alpha^+ = \lim_{m\to\infty} \alpha_m = \bar{\alpha}$. Then

$$P(T, \bar{\alpha} \log|T'(x)|) = \log s^2.$$

Finally, since, as we have already said, $\lim_{m\to\infty} \mathrm{HD}(\mu_m) = \mathrm{HD}(\mu)$, we have $\bar{\alpha} \leq \mathrm{HD}(\mu)$. The same arguments show point (iii) of corollary 4.4: it is only sufficient to replace (4.10) for linear mappings, with the Bowen-Ruelle formula $P(L_m, -d_{\mathrm{H}}^m \log|L'_m(x)|) = 0$.

Inspecting condition (ii) of corollary 4.4, one would expect that $\tilde{\alpha}=\alpha$, where α is the divergence abscissa for T. We are only able to prove that the following theorem is true.

Theorem 4.5. $\alpha \leqslant \tilde{\alpha}$. As a consequence of the proof we also have that the approximant energy integrals $\int_{C_m}\int_{C_m}|x-y|^{-\delta}\,\mathrm{d}\mu_m(x)\,\mathrm{d}\mu_m(y)$ converge pointwise for $\delta<\alpha$ to the energy integral for the map T with respect to the balanced measure.

Proof. We suppose, in contrast, that $\alpha>\tilde{\alpha}$ and consider a value $\tilde{\tilde{\alpha}}$ with $\tilde{\alpha}<\tilde{\tilde{\alpha}}<\alpha$ in such a way that the double integral $\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\tilde{\tilde{\alpha}}}$ is well defined. To study this integral, we apply formula (4.2). We begin to define the functions on the set $Q=\{(x,y)\in\mathbb{R}^2;\ 0<|x-y|\leqslant 1\}$:

$$F_n(x,y)=\begin{cases}|x-y|^{-\tilde{\tilde{\alpha}}} & |x-y|\geqslant 1/n\\ n^{\tilde{\tilde{\alpha}}} & |x-y|<1/n.\end{cases}$$

We clearly have

$$\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\tilde{\tilde{\alpha}}}=\lim_{n\to\infty}\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)F_n(x,y).$$

In the integral on the right-hand side we apply formula (4.2) twice to a power $T^m_{|V}$ of the map, $m\geqslant 1$, namely it gives

$$\lim_{n\to\infty}\lim_{j\to\infty}\left(\frac{1}{s^m}\right)^j\sum_{\xi_j\in(T^m)^{-j}\{\xi_0\}}\left(\lim_{k\to\infty}\left(\frac{1}{s^m}\right)^k\widetilde{\sum_{\xi_k\in(T^m)^{-k}\{\xi_0\}}}|\xi_j-\xi_k|^{-\tilde{\tilde{\alpha}}}\right)$$
$$+\lim_{n\to\infty}\lim_{j\to\infty}\left(\frac{1}{s^m}\right)^j\sum_{\xi_j\in(T^m)^{-j}\{\xi_0\}}\left[\lim_{k\to\infty}\left(\frac{1}{s^m}\right)^k n^{\tilde{\tilde{\alpha}}}N_n(k,\xi_j)\right]$$

where $\{\xi_0\}$ is any point in $[0,1]$, the tilde on the series means that we only sum on the ξ_k that do not belong to the open ball of centre ξ_j and radius $1/n$, and $N_n(k,\xi_j)$ is the cardinality of the set $\Omega(k,\xi_j)\equiv\{\xi_k\in(T^m)^{-k}\{\xi_0\};\ |\xi_k-\xi_j|<1/n\}$.

If we now put $\bar{\xi}_j$ and $\bar{\xi}_k$, two elements, respectively, of the sets $(L_m)^{-j}\{\xi_0\}$ and $(L_m)^{-k}\{\xi_0\}$, such that each pair $(\xi_j,\bar{\xi}_j)$ and $(\xi_k,\bar{\xi}_k)$ belongs to the same element of the collection of the s^m disjoint sets given by $T^{-m}[0,1]$, we clearly have

$$|\xi_j-\xi_k|\leqslant 2\Delta_m+|\bar{\xi}_j-\bar{\xi}_k|$$

where $\Delta_m=\operatorname{diam}\{T^{-m}[0,1]\}$. We call 'corresponding' the elements of a couple $(\xi_j,\bar{\xi}_j)$. We now fix ξ_j and consider the points $\xi_k\in\Omega(k,\xi_j)$. If we sum on the corresponding elements, we get

$$\sum_{\bar{\xi}_k\in(L_m)^{-k}\{\xi_0\}}(2\Delta_m+|\bar{\xi}_j-\bar{\xi}_k|)^{-\tilde{\tilde{\alpha}}}\leqslant\sum_{\bar{\xi}_k\in(L_m)^{-k}\{\xi_0\}}(\Delta_m)^{-\tilde{\tilde{\alpha}}}\leqslant N_n(k,\xi_j)(\Delta_m)^{-\tilde{\tilde{\alpha}}}\leqslant n^{\tilde{\tilde{\alpha}}}N_n(k,\xi_j).$$

The last inequality is justified if we put $n\geqslant 1/\Delta_m$ and this can certainly be satisfied because we are working at m fixed and the limit in n comes before this. In conclusion we have

$$\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\tilde{\tilde{\alpha}}}\geqslant\int_{C_m}\int_{C_m}\mathrm{d}\mu_m(x)\,\mathrm{d}\mu_m(y)(|x-y|+2\Delta_m)^{-\tilde{\tilde{\alpha}}}.$$

If we now fix $\varepsilon>0$, there is a number $m_\varepsilon>0$ such that, for every $m>m_\varepsilon$, we have

$$\int_J\int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\tilde{\tilde{\alpha}}}\geqslant\int_{C_m}\int_{C_m}\mathrm{d}\mu_m(x)\,\mathrm{d}\mu_m(y)(|x-y|+\varepsilon)^{-\tilde{\tilde{\alpha}}}.$$

Taking the lim sup for $m \to +\infty$, and since ε is arbitrary, we get

$$\int_J \int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\tilde{\alpha}} \geqslant \overline{\lim_{m\to\infty}} \int_{C_m} \int_{C_m} |x-y|^{-\tilde{\alpha}}\,\mathrm{d}\mu_m(x)\,\mathrm{d}\mu_m(y) = +\infty$$

since, for $m \to \infty$, only a finite number of the divergence abscissae for the linear Cantor sets stay on the right of $\tilde{\alpha}$. Thus $\alpha \leqslant \tilde{\alpha}$. When $\delta < \alpha$, the energy integral for the non-linear map T is convergent; if we repeat the above arguments interchanging the maps L_m and T^m, we get

$$\begin{aligned}\int_J \int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)|x-y|^{-\delta} &\geqslant \int_{C_m} \int_{C_m} \mathrm{d}\mu_m(x)\,\mathrm{d}\mu_m(y)(|x-y|+2\Delta_m)^{-\delta} \\ &\geqslant \int_J \int_J \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)(|x-y|+4\Delta_m)^{-\delta}.\end{aligned}$$

Taking the limit for $m \to \infty$ we obtain the last assertion of the theorem.

Remark 4.6. Given a non-linear Cantorian mapping it is very easy to compute numerically the sets $T_i^{-m}[0, 1]$ and all the variables for the linear mapping through (4.4)-(4.6) and (4.9), also for large m. In [19] we have applied this method to the quadratic map $z' = z^2 - p$, $p > 2$, whose repeller is a totally disconnected Julia set, and we have shown that the convergence of the various quantities given by corollary 4.4 is rather fast and quite good; moreover the method can be easily extended to conformal mappings in the plane. However a question remains open: referring to theorem 4.5, is $\alpha = \tilde{\alpha}$? Is formula (4.10) true also for a non-disconnected mixing repeller?

Remark 4.7. Inspecting the left-hand side of the limit in theorem 4.3, we recognise in it the thermodynamic limit of the partition function:

$$Z_m(\delta) = \sum_{A \in \mathscr{A}^m} |A|^{-\delta}$$

where $|A| = \operatorname{diam} A$ and, as usual,

$$\mathscr{A}^m = \bigvee_{i=0}^{m} T^{-i}\mathscr{A}^0$$

$\mathscr{A}^0$ being the partition: $\mathscr{A}^0 = \{T_i^{-1}[0, 1] \cap J, i = 1, \ldots, s\}$. The same limit holds if we consider for a generic conformal mixing repeller J a Markov partition [25, 35] of J of sufficiently small diameter. More precisely we have this theorem.

Theorem 4.8. Let T be a C^1 conformal transformation of degree s defined in an open neighbourhood of the mixing repeller J. For every $\varepsilon > 0$ there exists a $\chi(\varepsilon) > 0$ such that, if $\mathscr{M}^0$ is a Markov partition of J of diameter less than $\chi(\varepsilon)$, we have pointwise for every $\delta \in \mathbb{R}$:

$$\lim_{m\to\infty} \frac{1}{m} \log \sum_{M_\alpha^m \in \mathscr{M}^m} |M_\alpha^m|^{-\delta} = P(T, \delta \log\|D_x T\|)$$

where

$$\mathscr{M}^m = \bigvee_{k=0}^{m} T^{-k}\mathscr{M}^0.$$

Proof. We begin by observing that by theorem 2.2, fixed at $\varepsilon > 0$, there exists a number $\chi(\varepsilon) > 0$ such that, if A is a subset of J with $|A| < \chi(\varepsilon)$ and A_α^m is an element of $T^{-m}[A]$, $\alpha = 1, \ldots, s^m$, we have

$$\frac{e^{-m\varepsilon}|A|}{\|D_x T^m\|} \leqslant |A_\alpha^m| \leqslant \frac{e^{m\varepsilon}\|A\|}{\|D_x T^m\|}$$

where x is any point in A_α^m. Using this fact we have (we suppose here $\delta > 0$; for the other case, similar bounds apply; moreover, we neglect the finite terms when the limit for $m \to \infty$ is performed):

$$-\varepsilon\delta + \frac{1}{m}\log \sum_{\substack{x \in M_\alpha^m \\ M_\alpha^m \in \mathcal{M}^m}} \|D_x T^m\|^\delta \leqslant \frac{1}{m}\log \sum_{M_\alpha^m \in \mathcal{M}^m} |M^m|^{-\delta} \leqslant \varepsilon\delta + \frac{1}{m}\log \sum_{\substack{x \in M_\alpha^m \\ M_\alpha^m \in \mathcal{M}^m}} \|D_x T^m\|^\delta.$$

Now we consider the lower bound and compute $\|D_x T^m\|^\delta$ at the point $x \in M_\alpha^m$ where it attains the maximum. Then the quantity

$$\frac{1}{m}\log \sum_{\substack{x \in M_\alpha^m \\ M_\alpha^m \in \mathcal{M}^m}} \|D_x T^m\|^\delta$$

coverges to the 'pressure' $P_c(T, \delta\log\|D_x T\|)$ constructed on a closed cover of J (see [26], § 3). But, being $\mathcal{M}^0$ a μ_δ-disjoint closed cover of J, where μ_δ is the (ergodic) equilibrium measure for the function $\delta \log\|D_x T\|$, by lemma 3.4 in [26] we can conclude that

$$P_c(T, \delta \log\|D_x T\|) \geqslant h(\mu_\delta) + \delta \int_J \log\|D_x T\| \, d\mu_\delta = P(T, \delta \log\|D_x T\|).$$

On the other hand, if $\tilde{\mathcal{M}}^0$ is an open cover of J of diameter less than $\chi(\varepsilon)$ such that each element $M_\alpha^0 \in \mathcal{M}^0$ is included in an element $\tilde{M}_\alpha^0$ of $\tilde{\mathcal{M}}^0$ and $\tilde{M}_\alpha^0 \cap \tilde{M}_\beta^0 = \varnothing$ when $M_\alpha^0 \cap M_\beta^0 = \varnothing$, given $M_\alpha^m \in \mathcal{M}^m$ it is easy to see, for the properties of the Markov partitions†, that there exists an open set $\tilde{M}_\alpha^m \in \tilde{\mathcal{M}}^m$, $M_\alpha^m \cap \tilde{M}_\alpha^m \neq \varnothing$, which is not covered by the other sets of $\tilde{\mathcal{M}}^m$.

Therefore, for each open subcover S of $\tilde{\mathcal{M}}^m$ we have, choosing the points x in the sets of type $\tilde{M}_\alpha^m$ described above,

$$\sum_{x \in M_\alpha^m \cap \tilde{M}_\alpha^m} \|D_x T^m\|^\delta \leqslant \inf_{S \subset \tilde{\mathcal{M}}^m} \left\{ \sum_{S^m \in S} \sup_{x \in S^m} \|D_x T^m\|^\delta \right\}.$$

Taking the logarithm and dividing by m, we have that the right-hand side tends to the pressure $P(T, \delta \log\|D_x T\|)$ when $m \to +\infty$ (see § 3). Combining this with the preceding result for the lower bound, and since ε can be chosen arbitrarily small, we obtain the desired result.

5. Generalised dimensions

Recently an infinite number of dimensions have been introduced to characterise the invariant strange set J [20]. In this section we propose a new definition of the spectrum of generalised dimensions which is independent of the measurable partition of the set and which leads, if referred to the subsets of full measure, to define the generalised

† Here we essentially use the facts that, if $\{M_1, \ldots, M_p\}$ is a Markov partition, then $T(\bigcup_i \partial M_i) \subset \bigcup_i \partial M_i$, where ∂M_i is the boundary of M_i and for each element $M_{l,i}^{-1}$ of the set $T^{-1}[M_i]$, $i = 1, \ldots, p$, $l = 1, \ldots, s$, there exists q such that $M_{l,i}^{-1} \subset M_q$ (see [35]).

dimensions of the measure that are the natural extension of the Hausdorff dimension of the measure (see § 2).

Actually some of our results apply to all the compact Riemannian manifold for which the limit

$$\lim_{l\to 0^+} \frac{\log \mu(B(x, l))}{\log l} = \mathrm{HD}(\mu) = \beta \tag{5.1}$$

exists μ almost everywhere (see § 2 for the mathematical notation). As well as for conformal mixing repellers, the existence of the limit (5.1) has been proved for a class of non-uniformly expanding maps of the interval [15], for the Lozi map [36], and for C^2 diffeomorphisms of compact surfaces [16]: in this case, if μ is an ergodic probability measure with μ Lyapunov exponents $\Lambda_1 > 0 > \Lambda_2$ then

$$\beta = h(\mu)\left(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right). \tag{5.2}$$

The definition of the generalised dimensions given in [20] starts by considering a compact subset J of a Riemannian compact manifold M, of μ measure 1, where μ is a probability measure on the Borel subsets of M. Then one takes a countable partition of J by μ measurable sets $\{A_k\}_{k=1}^{\infty}$ of diameter less than ε and define the partition function, for real q and τ (the reference to the partition $\{A_k\}$ is dropped),

$$\Gamma(q, \tau) = \sum_{k=1}^{\infty} \frac{\mu(A_k)^q}{(\mathrm{diam}\, A_k)^{\tau}}. \tag{5.3}$$

When $\varepsilon \to 0$, it was argued that $\Gamma(q, \tau)$ goes to infinity for $\tau > \tau(q)$ and to zero for $\tau < \tau(q)$. This allows us to define the generalised dimensions D_q as

$$D_q(q-1) = \tau(q). \tag{5.4}$$

Just using this definition, applied to conformal mixing repellers and for certain partitions of the set, it is possible to give a complete characterisation of the generalised dimensions computed with respect to the equilibrium measures of the pressure (the Gibbs measure). We return to this point below; here, we want to stress the dependence of the dimensions on the measure supported by the set.

The definition of the partition function (5.3) is not satisfactory, because it does not permit us to verify the existence of the various limits involved in it. So we slightly change the definition in such a way as to transform the partition function into a measure on the Borel subset of M. To do this we begin by putting

$$H^{\tau}_{q,l}(J) = \inf_{B_k(x)\in \mathscr{B}_l} \sum_{k=1}^{\infty} \frac{\mu[B_k(x)]^q}{[\mathrm{diam}\, B_k(x)]^{\tau}} \tag{5.5}$$

where the infimum is over all countable coverings $\mathscr{B}_l = \{B_k(x)\}_{k=1}^{\infty}$ of a subset $J \subset M$, by closed balls $B_k(x)$ of centre $x \in J$ and of diameter less than l. As before, μ is a probability measure on the Borel subsets of M. When l decreases, the infimum is taken over a smaller set of coverings. Then $H^{\tau}_{q,l}(J)$ is a non-decreasing function of l when $l \to 0^+$ and the following (possibly infinite) limit exists:

$$\lim_{l\to 0^+} H^{\tau}_{q,l}(J) = H^{\tau}_{q}(J). \tag{5.6}$$

By standard arguments similar to those which define the Hausdorff measures [23], it is straightforward to verify that $H^{\tau}_{q}(J)$ is, as a function of J, a metric outer measure, so that it becomes a measure on the Borel subsets of M [29]. From now on, $J \subset M$ will be a subset of this type. Since

$$
\begin{aligned}
H^{\tau}_{q,l}(J) &= \inf_{\mathscr{B}_l} \sum_{k=1}^{\infty} \frac{\mu[B_k(x)]^q}{[\operatorname{diam} B_k(x)]^{\tau}} \\
&= \inf_{\mathscr{B}_l} \sum_{k=1}^{\infty} \frac{\mu[B_k(x)]^q}{[\operatorname{diam} B_k(x)]^{\tau'}} \frac{1}{[\operatorname{diam} B_k(x)]^{\tau-\tau'}} \\
&\geqslant \frac{1}{l^{\tau-\tau'}} H^{\tau'}_{q,l}(J) \qquad \text{for} \qquad \tau' < \tau
\end{aligned}
$$

we have that, for fixed q, when $H^{\tau'}_q(J)$ is positive then $H^{\tau}_q(J)$ is infinite, and when $H^{\tau}_q(J)$ is finite, $H^{\tau'}_q(J)$ must be zero. Thus there exists a unique point $\tau_{q,\mu}(J)$ such that $H^{\tau}_q(J)$ is zero when $\tau < \tau_{q,\mu}(J)$ and $H^{\tau}_q(J)$ is ∞ when $\tau > \tau_{q,\mu}(J)$.

To be correct one could have, for fixed q and for all $\tau \in \mathbb{R}$, $H^{\tau}_q(J) = +\infty$ or $H^{\tau}_q(J) = 0$. In the first case $\tau_{q,\mu}(J) = -\infty$ and in the second one $\tau_{q,\mu}(J) = +\infty$. But, since in a Euclidean space the Hausdorff dimension of any set is finite, it is easy to see that $\tau_{q,\mu}(J) \neq -\infty$ for $q > 0$ and $\tau_{q,\mu}(J) \neq +\infty$ for $q < 0$. However, these extreme cases do not happen for the systems which satisfy (5.1) and (5.15) as a consequence of theorem 5.2 and proposition 5.3 quoted below.

We call $\tau_{q,\mu}(J)$ the (q, μ) Hausdorff index of J, remarking on the fact that it depends both on q and the defining measure μ. Clearly the measure $H^{\tau}_q(J)$ evaluated for $\tau = \tau_{q,\mu}(J)$ may be 0, finite or infinite; moreover, it is easy to show that, for fixed q, $\tau_{q,\mu}(J') \geqslant \tau_{q,\mu}(J)$ when $J' \subset J$. Since we are thinking of J as an invariant set for some transformation on M, from now on we suppose that J be a Borel subset of M and $\mu(J) = 1$. Then, in analogy with (5.3), we define the generalised dimensions of J as

$$D_{q,\mu}(J)(q-1) = \tau_{q,\mu}(J). \tag{5.7}$$

By the monotony property of $\tau_{q,\mu}(J)$ as a set function we have

$$D_{q,\mu}(J) \leqslant D_{q,\mu}(J') \qquad \text{when } J' \subset J \text{ and } q > 1 \tag{5.8}$$

$$D_{q,\mu}(J) \geqslant D_{q,\mu}(J') \qquad \text{when } J' \subset J \text{ and } q < 1. \tag{5.9}$$

For $q = 0$ we clearly obtain $D_{0,\mu}(J) = d_{\mathrm{H}}$, the Hausdorff dimension of J. To make this more precise the relationship between the Hausdorff dimension and the Hausdorff index, and motivated by the inequalities (5.8) and (5.9), it is natural to put the following.

Definition 5.1. Let μ a Borel probability measure on M. We define the generalised dimensions of the measure μ as

$$\underline{D}_q(\mu) = \inf_{\substack{Y \subset M \\ \mu(Y)=1}} D_{q,\mu}(Y) \qquad \text{when } q \leqslant 1 \tag{5.10}$$

$$\bar{D}_q(\mu) = \sup_{\substack{Y \subset M \\ \mu(Y)=1}} D_{q,\mu}(Y) \qquad \text{when } q \geqslant 1. \tag{5.11}$$

Theorem 5.2. In the hypothesis of definition 5.1 and the existence of the limit (5.1) we have

$$\underline{D}_q(\mu) = \bar{D}_q(\mu) = \beta \qquad \forall q \in \mathbb{R}. \tag{5.12}$$

Proof. It will be sufficient to show that, for every μ measurable subset $\bar{J} \subset M$ where the limit (5.1) holds everywhere, we have $\beta q - \tau_{q,\mu}(\bar{J}) = \beta$, $q \in \mathbb{R}_-$.

Let us substitute the continuous variable l with a descending sequence $l_n \to 0$, $n \to \infty$, then fix a sequence $\varepsilon_k \to 0$, $k \to \infty$. By the proof of Egorov's theorem (see [37]), we can construct an increasing sequence of μ measurable sets $\{J_k\}_{k=1}^{\infty}$, $J_k \subset \bar{J}$, of measure $\mu(J_k) > 1 - \varepsilon_k$ where the convergence in (5.1) is uniform for $n \to \infty$ and such that

$$\lim_{k\to\infty} \mu(J_k) = \mu(\bar{J}) = 1.$$

Now we compute $H^{\tau}_{q,l_n}(J_k)$ covering J_k with closed balls $\{B_j(k, x)\}_{j=1}^{\infty}$, $x \in J_k$, and whose diameter is less than l_n. Then, for every $\delta > 0$, there is an $n_\delta > 0$ such that, uniformly on J_k and up to a finite constant deriving from the comparison between the diameter of the ball and its radius,

$$\inf_{B_j \in \mathscr{B}_{l_n}} \sum_{j=1}^{\infty} [\operatorname{diam} B_j(k, x)]^{(\beta+\delta)q-\tau} \leq H^{\tau}_{q,l_n}(J_k)$$

$$\leq \inf_{B_j \in \mathscr{B}_{l_n}} \sum_{j=1}^{\infty} [\operatorname{diam} B_j(k, x))]^{(\beta-\delta)q-\tau}$$

for $n > n_\delta$ and $q \geq 0$; for $q < 0$ similar arguments apply†. Taking first the limit for $n \to +\infty$ and since δ is arbitrary, we get

$$H^{\tau}_q(J_k) = K_{\beta q-\tau}(J_k) \tag{5.13}$$

where $K_{\beta q-\tau}(J_k)$ is the $(\beta q - \tau)$ Hausdorff measure of J_k [29]. Starting from the fact that, for fixed q, $\tau' < \tau_{q,\mu}(J_k)$ implies that $\beta q - \tau' > d_{\mathrm{H}}(J_k)$ and $\tau' > \tau_{q,\mu}(J_k)$ implies that $\beta q - \tau' < d_{\mathrm{H}}(J_k)$, where $d_{\mathrm{H}}(J_k)$ is the Hausdorff dimension of J_k, it is easy to see that

$$\beta q - \tau_{q,\mu}(J_k) = d_{\mathrm{H}}(J_k). \tag{5.14}$$

Since, for every $k \geq 1$ and q fixed, we have

$$\tau_{q,\mu}(J_k) \geq \tau_{q,\mu}(J_{k+1})$$

then

$$\lim_{k\to\infty} \tau_{q,\mu}(J_k) = \bar{\tau}_q \geq \tau_{q,\mu}(\bar{J}).$$

On the other hand, we have

$$\lim_{k\to\infty} H^{\bar{\tau}_q}_q(J_k) = H^{\bar{\tau}_q}_q(\bar{J}) = 0$$

from which

$$\tau_{q,\mu}(\bar{J}) \geq \bar{\tau}_q.$$

Then $\bar{\tau}_q = \tau_{q,\mu}(\bar{J})$.

A similar argument applies to the Hausdorff dimension. Thus we finally get $\beta q - \tau_{q,\mu}(\bar{J}) = d_{\mathrm{H}}(\bar{J})$. By the very choice of $\bar{J}$ and by proposition 2.1 in [16], we have $d_{\mathrm{H}}(\bar{J}) = \beta$. This finishes the proof.

† To be correct, we have to fix the non-increasing sequence l_n in such a way that $\log l_{n+1}/\log l_n \to 1$, as in § 2. Then one can show that, for every $\delta > 0$, there exists a $n_\delta > 0$ such that for every $l < l_{n_\delta}$ one has uniformly on $x \in J_k$

$$\frac{\beta-\delta}{1+\delta} \leq \frac{\log \mu(B_J(x, l))}{\log l} \leq (\beta+\delta)(1+\delta).$$

Replacing the constants $(\beta-\delta)$ and $(\beta+\delta)$ with $(\beta-\delta)/(1+\delta)$ and $(\beta+\delta)(1+\delta)$, respectively, in the inequalities before (5.13), we can continue the proof in the same manner.

Also in this context, the Hausdorff dimension of the measure plays a fundamental role in the characterisation of the fractal geometry of a set. When the limit (5.1) does not exist, following the same ideas of theorem 5.2 it is not difficult to prove the following proposition.

Proposition 5.3. Suppose that μ almost everywhere $x \in M$ there exist two finite constants $\underline{\beta}$ and $\bar{\beta}$ such that uniformly in x

$$\underline{\beta} \leqslant \limsup_{l \to 0^+} \frac{\log \mu(B(x, l))}{\log l} \leqslant \limsup_{l \to 0^+} \frac{\log \mu(B(x, l))}{\log l} \leqslant \bar{\beta}.$$

Then

$$\underline{\beta} \leqslant \underline{D}_q(\mu) \leqslant \bar{\beta} \qquad q \leqslant 0 \tag{5.15}$$

$$\frac{q\bar{\beta} - \underline{\beta}}{q-1} \leqslant \underline{D}_q(\mu) \leqslant \frac{q\underline{\beta} - \bar{\beta}}{q-1} \qquad 0 \leqslant q \leqslant 1 \tag{5.16a}$$

$$\frac{q\underline{\beta} - \bar{\beta}}{q-1} \leqslant \bar{D}_q(\mu) \leqslant \frac{q\bar{\beta} - \underline{\beta}}{q-1} \qquad q \geqslant 1. \tag{5.16b}$$

When $q \to +\infty$, the last relation simply becomes

$$\underline{\beta} \leqslant \bar{D}_q(\mu) \leqslant \bar{\beta} \qquad q \to +\infty.$$

As we have already seen, for some dynamical systems, such as the C^2 diffeomorphisms of a compact surface or the mixing repellers, the bounds $\underline{\beta}$ and $\bar{\beta}$ are exactly known.

Now we leave the set function (5.5) and return to the partition function (5.3) and consider a Markov partition of $\mathcal{M}^0$ of a conformal mixing repeller J invariant under a mapping T of degree s. An easy generalisation of theorem 4.1 allows us to endow J with a balanced measure μ_{B}. If we rewrite the partition function $\Gamma(q, \tau)$ as (we use here the notation of theorem 4.6):

$$\Gamma_n(q, \tau) = \sum_{M_\alpha^n \in \mathcal{M}^n} \frac{\mu_{\mathrm{B}}(M_\alpha^n)^q}{|M_\alpha^n|^\tau} \tag{5.17}$$

we can prove that, when $n \to +\infty$, the changeover point $\tau(q)$ is related in a simple manner to the topological pressure. More precisely we have this proposition.

Proposition 5.4. Given a conformal mixing repeller of a transformation T of degree s and the balanced measure on it, for every $q \in \mathbb{R}$, the changeover point $\tau(q)$ is the unique solution of the equation:

$$P(T, \tau(q) \log\|D_x T\|) = q \log s = q h_{\mathrm{top}}. \tag{5.18}$$

Proof. The proof is an easy consequence of theorem 4.6 and the fact that the balanced measure of an atom $M_\alpha^n \in \mathcal{M}^n$ is simply $s^{-(n+1)}$ times the measure of an atom of $\mathcal{M}^0$.

A considerable improvement of proposition 5.4 is in [38] where we are able to compute the spectrum of the generalised dimensions with respect to the equilibrium measures (denoted now as μ_β) for the function: $-\beta \log\|D_x T\|$, $\beta \in \mathbb{R}$. More precisely, using the Walter theory of the Ruelle–Perron–Frobenius operator [39], we bound the μ_β measure of an atom M_α^n of a Markov partition, uniformly in n and on J. This is the central step to obtain proposition 5.5.

Proposition 5.5. In the hypothesis of proposition 5.4, and if we put on the repeller the equilibrium measure μ_β, the relative generalised dimensions $D_q^\beta(J)$ satisfy the equation:

$$P(T, [D_q^\beta(J)(q-1)-\beta q] \log\|D_xT\|) = qP(T, -\beta \log\|D_xT\|). \qquad (5.19)$$

For $\beta = 0$ we obtain proposition 5.4, while for $\beta = d_H$ we find that all the dimensions $D_q^{d_H}(J)$ are simply equal to the Hausdorff dimension, showing that the fractal appears uniform if 'observed' by this measure. We recall that the latter is equivalent to the d_H Hausdorff measure of J.

6. Concluding remarks

Results like (5.16) have been recently proposed in a few works [40-42]. They refer especially to disconnected invariant sets, where the role of the Markov partition is taken by the intervals which hierarchically construct the set (see, for example, the 'scales' of the Cantor sets described in § 4).

It is possible to extend the same formalism to non-hyperbolic sets. In this case, if $\|D_xT\| = 0$, for x belonging to the invariant set J, one cannot apply the topological definition of the pressure (§ 2) and hence the variational principle (3.2), but it may happen that J be the closure of the fixed points of T^n and also that the thermodynamic limit (3.3) exists. This is the case for the map $T:[-2, 2] \to [-2, 2]$, defined as $T(x) = x^2 - 2$. The analytic computation of the limit (3.3) for the function $\varphi = -\beta \log\|D_xT\|$, $\beta \in \mathbb{R}$, shows that the pressure, as a function of β, admits a phase transition for $\beta = -1$. A similar calculation can be performed for the tent map [43] and also in this case there is a phase transition for the pressure. Finally we briefly show how to extend the theory to axiom A attractors Λ in a two-dimensional manifold, which are locally the cartesian product of a Cantor set with an interval. We consider a Markov partition $\mathscr{A}^0$ of Λ and endow Λ with the physical measure (Sinai-Bowen-Ruelle measure [17]). The intersection of the stable side of each rectangle $A_\alpha^0 \in \mathscr{A}^0$ with Λ gives a partition of $W^s(x) \cap \Lambda$, where $\{W^s(x)\}$ denotes the stable foliation. Now we iterate $\mathscr{A}^0$: the stable sides of the A_α^0 dissected in an equal number of parts, say s, which we denote by $A_{\alpha_1}^s, \ldots, A_{\alpha_s}^s$. We suppose that the s scales $\lambda_1, \ldots, \lambda_s$ of this dissection and the conditional probabilities of the $A_{\alpha_i}^s$ (along the contracting subspaces) $p_1, \ldots, p_s$, are the same for all the rectangles and moreover, as a first approximation, that the process is self-similar for the successive iterates of $\mathscr{A}^0$. As a second approximation we use the s^2 scales deriving from the application of T^2 and so on, according to the scheme developed in theorem 4.3. The knowledge of the λ_i and p_i is sufficient to construct the partition function (5.15) and then to compute the generalised dimensions of the (Cantorian) intersection of Λ with the stable manifolds (the dimensions of the intersection of Λ with the unstable manifolds are all equal to 1, since the physical measure is smooth along the unstable subspaces).

In a future publication we will show how to numerically compute the λ_i and the p_i, by the direct calculation of the generalised Lyapunov exponents [42, 43]. The procedure can also be applied to non-hyperbolic attractors (Hénon map) to get a 'degree' of the non-hyperbolicity of the set. In the case of the generalised Baker transformation, the preceding calculations can be performed analytically, and we refer to [42] for the details.

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Note added. After this work was done, we became aware of a paper by Bohr and Rand [44], where a few results similar to those quoted in § 3 have been proved. Collecting theorem 6 in [44] with equation (4.11) we obtain another dynamical interpretation of the divergence abscissa α with respect to the balanced measure for linear Cantor set with s scales (and, possibly, for each non-linear Cantorian mapping with s inverses). We then have

$$h_{-\alpha} = \frac{\log s^2 - \alpha \rho_T}{1+\alpha}$$

where ρ_T is the theoretical escape rate and $h_{-\alpha}$ is the $(-\alpha)$ Renyi entropy with respect to the equilibrium measure for the function $-\log|L'(x)|$.

References

[1] Ruelle D 1982 *Ergodic Theor. Dyn. Syst.* **2** 99-107
[2] Kantz H and Grassberger P 1985 *Physica* **17D** 75-86
[3] Takesue S 1986 *Prog. Theor. Phys.* **76** 11-22
[4] Szèpfalusy P and Tel T 1986 *Phys. Rev.* A **34** 2520
[5] McDonald S, Grebogi C, Ott E and Yorke J A 1985 *Physica* **17D** 125-53
[6] Kadanoff L and Tang C 1984 *Proc. Natl Acad. Sci. USA* **81** 1276-9
[7] Widom M, Bensimon D, Kadanoff L and Shenker S J 1983 *J. Stat. Phys.* **32** 443-54
[8] Sullivan D 1983 *Geometric Dynamics* (*Lecture Notes in Mathematics* **1007**) (Berlin: Springer) p 320
[9] Bessis D, Servizi G, Turchetti G and Vaienti S 1987 *Phys. Lett.* **119A** 345-7
[10] Tel T 1986 *Escape rate from strange sets as an eigenvalue. Preprint* ICTP, Trieste
[11] Pelikan S 1985 *Trans. Am. Math. Soc.* **292** 695-703
[12] Barnsley M F and Demko S 1985 *Proc. R. Soc.* A **399** 243-75
[13] Manning A 1984 *Ann. Math.* **119** 425-30
[14] Ledrappier F 1984 *C. R. Acad. Sci., Paris* **299** 37-40
[15] Ledrappier F and Misiurewicz M 1985 *Ergodic Theor. Dyn. Syst.* **5** 595-610
[16] Young L S 1982 *Ergodic Theor. Dyn. Syst.* **2** 109-24
[17] Eckmann J P and Ruelle D 1985 *Rev. Mod. Phys.* **57** 617
[18] Turchetti G and Vaienti S 1989 *Egypt. J. Phys.* to be published
[19] Turchetti G and Vaienti S 1987 *Phys. Lett.* to be published
[20] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev.* A **33** 1141
[21] Ledrappier F 1981 *Commun. Math. Phys.* **81** 229-38
[22] Brin M and Katok A 1983 *Geometric Dynamics* (*Lecture Notes in Mathematics* **1007**) (Berlin: Springer) p 182
[23] Billingsley P 1965 *Ergodic Theory and Information* (New York: Wiley)
[24] Ruelle D 1979 *Publ. Math. IHES* **50** 275-306
[25] Ruelle D 1978 *Thermodynamic Formalism* (New York: Addison-Wesley)
[26] Walters P 1975 *Am. J. Math.* **97** 937-71
[27] Ljubich M Ju 1983 *Ergodic Theor. Dyn. Syst.* **3** 351-85
[28] Brolin H 1965 *Ark. Math.* **6** 103-44
[29] Falconer K J 1985 *The Geometry of Fractal Sets* (Cambridge: Cambridge University Press)
[30] Vaienti S 1987 *Nuovo Cimento* B **99** 77
[31] Landkof N S 1972 *Foundations of Modern Potential Theory* (Berlin: Springer)
[32] Lopes A 1986 *A note on δ-equilibrium measures for rational maps. Preprint* Porto Alegre

[33] Bessis D, Fournier J, Servizi G, Turchetti G and Vaienti S 1987 *Phys. Rev.* A **36** 920
Vaienti S 1987 *Proc. Conf. on Dynamic Systems, Rome, 1986* (Singapore: World Scientific) p 47
[34] Grassberger P and Procaccia I 1983 *Physica* **9D** 189
[35] Krzyzewski K 1971 *Bull. Acad. Pol. Sci.* **4** 291-3
[36] Collet P and Levy Y 1984 *Commun. Math. Phys.* **93** 461
[37] Halmos P 1964 *Measure Theory* (New York: Van Nostrand)
[38] Vaienti S 1988 *J. Phys. A: Math. Gen.* **21** 2313-20
[39] Walters P 1978 *Trans. Am. Math. Soc.* **236** 121
[40] Feigenbaum M J 1987 *J. Stat. Phys.* **46** 919
[41] Collet P, Lebowitz J L and Porzio A 1987 *J. Stat. Phys.* **47** 609
[42] Bessis D, Paladin G, Turchetti G and Vaienti S 1988 *J. Stat. Phys.* to be published
[43] Benzi R, Paladin G, Parisi G and Vulpiani A 1985 *J. Phys. A: Math. Gen.* **18** 2157
[44] Bohr T and Rand D 1987 *Physica* **25D** 387-98